

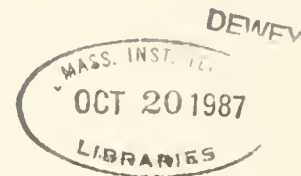


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The Behavior of the Term Structure of Interest Rates
in a Real Business Cycle Model

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and

Rajnish Mehra***

WP#1912-87

Revised June 1987

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1. Introduction

This paper follows the tradition of real business cycle (RBC) theory as initiated by Kydland and Prescott (1982). A basic premise of this line of research is the view that aggregate macroeconomic models should be evaluated primarily with regard to their ability to replicate observed empirical regularities. Thus far, attention has focused principally on the ability of this class of models to approximate, satisfactorily, the matrix of variances and covariances of macroeconomic aggregates.

Our work is an attempt to extend this modelling perspective to explain observed regularities in the financial markets. In particular we seek to determine the extent to which a very simple, reduced form RBC model is able to account for properties of the term structure of interest rates. Our specific context for this exercise is the simple one-good neoclassical stochastic growth model.¹ This choice can be justified in a number of ways. (1) While it is by no means the most sophisticated construct available (see, e.g., Kydland and Prescott (1982)) and, indeed, does not in fact provide a particularly good 'match' for aggregate data (see Lucas (1985)) the neoclassical model is nevertheless the underlying foundation of all RBC models and for this reason it seems the appropriate place at which such a study as ours should begin. Furthermore, its 'stripped down' simplicity is likely to afford the most direct enrichment of intuition. (2) For the neoclassical model we can easily characterize the Markov processes on the real state variables (consumption, investment, and output) and compute their stationary distributions. This, in turn, allows us easily to derive the exact probability distribution of prices of bonds of various maturities. From this data, it is a simple

matter to calculate the various rates of return, forward premia, and price and rate autocorrelations critical to our study. (3) In our consideration of bond market informational and allocative efficiencies, we intend to compare results derived from this model with those obtained from a variety of econometric studies, where stationarity of the relevant time series is typically assumed. We therefore focus on the model's stationary equilibria. For this case, work by Prescott and Mehra (1980) has shown that the optimal stationary allocations arising from this model may be regarded as the competitive equilibrium of a homogeneous consumer economy in recursive equilibrium.² Thus, we are assured that our equilibrium security prices are perfectly consistent both with the real and financial sides of the economy.

Now the term structure literature is clearly so vast that it would be highly presumptuous to attempt to evaluate the model -- even qualitatively -- in the context of all its strands.³ Thus we restrict our attention to three specific areas. First, we examine the changing shape of the yield curve over our artificial economy's "business cycle." Second, we employ the model to study various implications of informational and allocative efficiency -- properties our artificial economy must, by assumption, possess. We find, for example, that long term rates are less volatile than short term rates (a fact supported by the empirical literature) and that holding premia can be highly correlated over time. Third, we evaluate the accuracy of forward rates as estimators of future spot rates -- a traditional issue in the term structure literature -- in our model context. These results are also then compared with what is found in the empirical literature. Lastly, we study the effects of shifts in the economy's underlying parameters on the yield curve. Among

the issues we consider in this context are the effects of (i) changes in the structure of uncertainty in the economic environment and (ii) changes in time preferences and attitudes toward risk of the market participants on the term structure.

There is a cost to everything, however, and our model setting is not without its drawbacks. In particular, the high level of generality makes it exceedingly difficult to derive significant results without imposing restrictions on preferences and technology. Thus we primarily undertake a numerical examination of the model for a wide class of parameter values and offer intuitive explanations for the results obtained. While this numerical analysis is in the RBC tradition, our objectives are somewhat more limited than what is typically sought in that literature. For certain aspects of the term structure we shall be concerned only to what extent this model explains qualitatively observed reality. Thus we are at times less constrained as to our choices of parameter values than a formal RBC study would allow.

An outline of the paper is as follows: Section 2 reviews the basic model and derives expressions for the pricing of discount bonds of various maturities. In Section 3 (and Appendix 2) we present an outline of the numerical procedures employed. Section 4 details the "shape" of the yield curve while Section 5 considers certain implications of market efficiency. Section 6 explores the information content of the yield curve and Section 7 contains "comparative dynamics" results. This is followed by a brief summary.

2. The Economy

2.1 Model Overview

Consider the central planning dynamic stochastic growth problem

$$(P) \quad \max_{\{c_t, k_{t+1}\}} E \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right)$$

$$\text{s.t. } c_t + k_{t+1} \leq f(k_t) \lambda_t, \quad k_0 \text{ given}$$

in which the representative agent's preferences are assumed time separable with period discount factor β and utility function $u(\cdot)$ defined over period consumption c_t . In this setting, it is customary to denote capital available for production in period t , by k_t ; $f(\cdot)$ thus represents the period technology which is subjected to the multiplicative stochastic factor λ_t and E denotes the expectations operator. The shock sequence $\{\lambda_t\}$, $0 < \underline{\lambda} \leq \lambda_t \leq \bar{\lambda} < \infty$ is assumed to follow a Markov process with transition density $F(d\lambda_{t+1}; \lambda_t)$ and cumulative distribution function $G(\cdot)$; that is,

$$G(\lambda_{t+1}^*) = \int_{\underline{\lambda}}^{\lambda_{t+1}^*} \int_{\underline{\lambda}}^{\bar{\lambda}} F(d\lambda_{t+1}; \lambda_t) G(d\lambda_t)$$

Problem (P) has been studied by Brock and Mirman (1972, 1973), and Donaldson and Mehra (1983), among others (see footnote 1). Under appropriate assumptions the basic results are two-fold: (1) optimal, time-invariant consumption $c(k, \lambda)$ and savings $s(k, \lambda)$ policy functions exist which solve (P) and (2) by the repeated application of these

policies, the economy converges to a well-defined steady state. This is summarized in Appendix 1.

Turning now to the pricing of bonds, we can introduce an implicit financial instruments market where a riskless asset is traded, this asset being in zero net supply. Since we are modelling a homogeneous agent economy, and, as the net demand for this asset must be zero in equilibrium, its existence will not affect the equilibrium. Its price is obtained in the usual way from the first order conditions of the representative consumer. See Mehra and Prescott (1985) for an illustration. In particular, given the current state (k_t, λ_t) , the equilibrium price $P = P(k_t, \lambda_t)$ of a one period "pure discount bond" (i.e., a security which unconditionally pays one unit of consumption at time $t + 1$) is therefore

$$(1) \quad P(k_t, \lambda_t) = \beta \int \frac{u'(c(k_{t+1}, \lambda_{t+1}))}{u'(c(k_t, \lambda_t))} F(d\lambda_{t+1}; \lambda_t)$$

An n-period pure discount bond would, by analogy, represent an unconditional promise to pay one unit of the consumption n periods hence. Any default free coupon bearing bond may be considered to be a portfolio of pure discount bonds. Our analysis can therefore be applied to price these bonds as well.

To price n-period pure discount bonds, it is necessary to consider the n-period transition function on the state variables. Denote the joint capital stock-shock conditional one-period distribution function by

$$H_1(k_{t+1}, \lambda_{t+1} \mid k_t, \lambda_t).$$

In this light, the two step transition function is defined by

$$(2) H_2(k_{t+2}, \lambda_{t+2} | k_t, \lambda_t) = \int H_1(k_{t+2}, \lambda_{t+2} | k_{t+1}, \lambda_{t+1}) H_1(dk_{t+1}, d\lambda_{t+1} | k_t, \lambda_t)$$

Since the state variables (k_t, λ_t) follow a joint stationary Markov process (Theorem 2.1) with strictly positive bounded support, and since $H_1(\cdot | \cdot)$ can be shown to be continuous (see Donaldson and Mehra (1983)), the equality defined by equation (2) makes sense. Continuing in a like manner, we can recursively define the n-period distribution function by

$$(3) \quad H_n(k_{t+n}, \lambda_{t+n} | k_t, \lambda_t) \\ = \int H_1(k_{t+n}, \lambda_{t+n} | k_{t+n-1}, \lambda_{t+n-1}) H_{n-1}(dk_{t+n-1}, d\lambda_{t+n-1} | k_t, \lambda_t)$$

The price of an n-period discount bond maturing in period $t+n$ can now be formulated:

$$(4) \quad P_n(k_t, \lambda_t) = \beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(k_t, \lambda_t))} H_n(dK_{t+n}, d\lambda_{t+n} | k_t, \lambda_t)$$

Given these prices, we have all the information necessary to calculate the term structure of interest rates. The model allows a multiplicity of consistent definitions of the term structure, of which we will consider the following two:

(i) The Conditional Term Structure: Given state (k_t, λ_t) , the conditional term structure $\{r_n(k_t, \lambda_t)\}$ satisfies:

$$(5) \quad r_n(k_t, \lambda_t) = \left[\frac{1}{P_n(k_t, \lambda_t)} \right]^{\frac{1}{n}} - 1$$

(ii) The Average Term Structure: The unconditional or average term structure $\{r_n\}$ would be computed as the average of the conditional rates:

$$(6) \quad r_n = \int \left[\frac{1}{P_n(k_t, \lambda_t)} \right]^{\frac{1}{n}} H(dk_t, d\lambda_t) - 1.^4$$

In what follows (especially Sections 5 and 6) we examine issues of the changing yield curve, market efficiency, and the predictive power of forward rates. These quantities can be calculated explicitly, given a knowledge of the probabilistic evolution of the economy. We do this in the context of a numerical simulation, since analytical results will, in many cases, depend upon the properties of the joint stationary distribution on (k, λ) , and upon the function governing state to state transitions. As a rule such properties are impossible to derive in a model of this generality. In the next section we briefly review the structure of this numerical exercise.

3. Numerical Simulation

For this exercise we explicitly solved problem (P) and computed the n -step transition functions $\{H_n(dk_{t+n}, d\lambda_{t+n} \mid k_t, \lambda_t)\}$, and the corresponding term structure of interest rates for the case of power preferences $u(c) = (c^\gamma - 1)/\gamma$, and Cobb-Douglas technology $f(k) = 2/3 k^\alpha \lambda$, $\alpha \in \{.25, .36, .5\}$. The parameter γ was chosen from the set $\{-2, -1, 0, .5, 1\}$, where $\gamma = 0$ corresponds to logarithmic utility, while β could assume values chosen from $\{.8, .9, .95, .96, .99\}$. The shock to technology λ was governed by a three-state Markov process with transition function $F(\lambda_{t+1}; \lambda_t)$, as described by the transition probability matrix

$$\lambda_t = \frac{1}{2} \quad \lambda_t = 1 \quad \lambda_t = \frac{3}{2}$$

$$\lambda_t = \frac{1}{2} \begin{vmatrix} \pi & (1-\pi)/2 & (1-\pi)/2 \\ (1-\pi)/2 & \pi & (1-\pi)/2 \\ (1-\pi)/2 & (1-\pi)/2 & \pi \end{vmatrix}$$

$$\lambda_t = 1 \begin{vmatrix} \pi & (1-\pi)/2 & (1-\pi)/2 \\ (1-\pi)/2 & \pi & (1-\pi)/2 \\ (1-\pi)/2 & (1-\pi)/2 & \pi \end{vmatrix}$$

$$\lambda_t = \frac{3}{2} \begin{vmatrix} \pi & (1-\pi)/2 & (1-\pi)/2 \\ (1-\pi)/2 & \pi & (1-\pi)/2 \\ (1-\pi)/2 & (1-\pi)/2 & \pi \end{vmatrix}$$

where $\pi \in \{.333, .5, .7, .9\}$.

Every choice of parameters α , γ , β , and π generates a range of possible values for the capital stock variable k . Our example was chosen so that for all parameter choices these values were constrained within the unit interval $[0,1]$. We utilized an equal length partition Γ of this interval with unit length .01.

In order to derive the term structure, it was first necessary to compute the optimal savings and consumption policies for (P). Here we followed the customary dynamic programming method of seeking a fixed point to the related functional equation, by a sequence of approximating iterations. Using these policy functions, we next generated the time series corresponding to the approximating stationary distribution. From this information the term structure can be calculated. This procedure is detailed in Appendix 2.

In what follows we blend theoretical results with intuition gained from the simulation just described. We turn first to a consideration of the shape of the yield curve and how this shape can vary as the economy evolves.

4. The Shape of the Yield Curve over the Cycle

This is a stationary economy. Each period, the asymptotic probability distribution on future states -- whether capital stock, consumption, or output -- is the same irrespective of the current state. Nevertheless, the effect of the random shock to technology coupled with the fact that capital stock is, in any period, the prior period's savings is to generate persistence (cyclical behavior) in the time series of capital stock, consumption, and output. We find, therefore, that the shape of the yield curve, as well, changes dramatically depending on the state of the economy relative to this cycle.⁵ These changes are summarized in the following three general observations (which hold for all parameter values). Here we identify the top and bottom of the cycle with, respectively, the maximum and minimum stationary capital stock-shock combinations (equivalently, maximum and minimum observed output).

Observation 4.1: The yield curve at the top of the cycle lies uniformly below the yield curve at the bottom of the cycle. The same relationship is generally observed also for states between these polar values.

Observation 4.2: The yield curve is rising at the top of the cycle and falling at the bottom of the cycle.

Observation 4.3: The average yield curve $\{r_n\}$ is upward sloping for all parameter choices.

These results have natural interpretations within the structure of this simple model. Turning first to Observation 4.1, we notice that consumption is particularly high at the top of the cycle, when output

achieves its highest level. Moreover, as a consequence of the ergodic property of the consumption series, but with less than perfect shock correlation, it is to be expected that consumption will, on average, be lower in the future. Thus, relatively low interest rates are sufficient to induce agents to shift consumption forward from the present to the future. At the bottom of the cycle consumption on average will rise as time passes, thus requiring relatively higher interest rates to induce agents to postpone consumption.

Related reasoning applies to Observation 4.2. The interest rates reflect, essentially, the average rate of growth in consumption over the respective future time periods. Looking to the future from the bottom of the cycle, the average growth rate will be positive but decreasing with the horizon. This initial high growth in consumption followed by progressively lower growth is perfectly reflected in the declining yield curve. Analogous reasoning explains the rising yield curve at the top of the cycle.

Observation 4.3 reflects the risk structure of our economy. Given an agent's current state, his probabilistic knowledge of the future states becomes less precise the further ahead he looks. The rising yield curve reflects the increasing risk premia necessary to induce agents to make financial commitments (i.e., the purchase of long bonds) in the face of increasing uncertainty. As will be further discussed in Section 6, this does not necessarily imply that future one period rates will exceed today's one period rate. This is consistent with the findings of e.g. Shiller et al. (1983) who conclude that implied forward rates provide poor forecasts of future spot rates.

At first appearance, Observations 4.1 and 4.2 appear inconsistent with generally accepted business cycle theory. A commonly held view seems to be that interest rates are procyclical and that the yield curve will be downward sloping at the top of the cycle (in the expectancy of lower future rates). These assertions typically apply to nominal rates, while in our model all rates are real (commodity) rates.

We are aware of only one empirical study which considers the behavior of real interest rates over the cycle, that of Prescott et al. (1983). These authors detect a negative lagged relationship between real interest rates and output levels. The contemporaneous relationship between output and real rates is also found to be negative, though not strongly significant. This issue, however, deserves further theoretical and empirical study. In particular, the role played by inventories in smoothing or possibly reversing this countercyclical behavior of interest rates must be addressed.

We close this section with a theoretical result which summarizes the asymptotic behavior of the conditional and average yield curves for this class of models. Its conclusions basically reflect the asymptotic probabilistic nature of the time series.

Theorem 4.1: (i) For any $(k, \lambda) \in \mathbb{R}^K \times [\underline{\lambda}, \bar{\lambda}]$, the conditional term structure is bounded above and below. (ii) Furthermore, for every (k, λ) , the asymptotic limit is $\frac{1}{\beta} - 1$. The average term structure thus has the same limit as well.

Proof: (i) For any $(k, \lambda) \in \mathbb{R}^K \times [\underline{\lambda}, \bar{\lambda}]$, the following inequalities are satisfied:

$$\begin{aligned} \beta^n \frac{u'(c(\bar{k}, \bar{\lambda}))}{u'(c(\underline{k}, \underline{\lambda}))} &\leq \beta^n \int \frac{u'(c(k_{t+n}, \lambda_{t+n}))}{u'(c(k, \lambda))} H(dk_{t+n}, d\lambda_{t+n} | k, \lambda) \\ &\leq \beta^n \frac{u'(c(\underline{k}, \underline{\lambda}))}{u'(c(\bar{k}, \bar{\lambda}))} \end{aligned}$$

Therefore,

$$\frac{1}{\beta} \left\{ \frac{1}{\frac{u'(c(\bar{k}, \bar{\lambda}))}{u'(c(\underline{k}, \underline{\lambda}))}} \right\}^{\frac{1}{n}} - 1 \leq r_n(k, \lambda) \leq \frac{1}{\beta} \left\{ \frac{1}{\frac{u'(c(\underline{k}, \underline{\lambda}))}{u'(c(\bar{k}, \bar{\lambda}))}} \right\}^{\frac{1}{n}} - 1$$

and hence

$$\frac{1}{\beta} \left\{ \frac{u'(c(\underline{k}, \underline{\lambda}))}{u'(c(\bar{k}, \bar{\lambda}))} \right\}^{\frac{1}{n}} - 1 \leq r_n(k, \lambda) \leq \frac{1}{\beta} \left\{ \frac{u'(c(\bar{k}, \bar{\lambda}))}{u'(c(\underline{k}, \underline{\lambda}))} \right\}^{\frac{1}{n}} - 1 \quad \blacksquare$$

(ii) From the equation above we see that both upper and lower bounds converge to $1/\beta - 1$ as $n \rightarrow \infty$. ■

5. Issues of Efficiency

This model is, by construction, informationally efficient (see Lucas (1978)). In this section we first examine the implications of this efficiency for the relative volatility of short and long term rates and one period holding returns and then compare these results with what has appeared in the literature. We next evaluate our model in light of alternative criteria of market efficiency that have been employed in the empirical literature. In doing this, we are not proposing to "judge" the validity of these criteria; rather, we are simply attempting to under-

stand better the qualitative legitimacy of our abstraction. It should be noted at the outset that most empirical studies are concerned with nominal interest rates, whereas this model computes only real interest rates. This is a substantial qualification to our results which we openly acknowledge. In order to make the discussions manageable, we need first to agree on a bit more related notation:

$HP_{t+j,n}^k$: the "annualized" holding period return over k periods on an n period maturity bond purchased j periods from the present.

$r_{t+j,n}$: yield to maturity on an n period bond purchased j periods from now.

f_{t+j}^k : k -period forward rate, j periods in the future

$FP_{t,n}^1$: $f_{t+n}^1 - E(r_{t+n,1})$, the forward premium.

Turning to the volatility issues, we first observe that in this model long term real rates vary less than short term real rates. Again, this may be seen as a consequence of the ergodic property of the consumption process, and is consistent with the predictions of other rational expectations models of nominal rates (see e.g. Shiller (1979, pp. 1190-1194)).

Observation 5.1: For all choices of parameter values, the standard deviation of the stationary distribution of the yield-to-maturity is seen to decline with maturity.

It is also frequently claimed that market efficiency requires that the volatility of one period nominal holding period returns on long bonds should be less than the volatility of short rates. Yet, this is not observed empirically (see, again, Shiller (1979)). Neither is it ob-

served in this model of real rates. To illustrate, Table (1) below compares the standard deviation of the one period holding return $HP_{t,n}^1$ on an n-period bond with that of the one period interest rate $r_{t,1}$ for a representative parameter set:

Table (1)

$$\alpha = .25, \beta = .95, \gamma = -1, \pi = .5$$

Standard Deviations

	$HP_{t,n}^1$	$r_{t,n}$
n = 1		.400
n = 2	.663	
n = 3	1.04	
n = 4	1.26	
n = 5	1.40	
n = 10	1.49	
n = 15	1.49	

Before examining the more traditional correlation tests of efficiency (price change correlations and the like), there is one more efficiency property to be considered. It has been asserted that in an efficient market successive holding period returns should be uncorrelated with any variables in agents' information sets and, in particular, with past holding period returns and rate spreads. Campbell and Shiller (1984), in particular, correlated today's excess one period holding return on long bonds above the short interest rate with the spread between the long and the short interest rate -- $\text{corr}[HP_{t,n}^1 - r_{t,1}, r_{t,n} - r_{t,1}]$.⁶ That is, what does the slope of the yield curve have to say about the ex post premium to holding long bonds? By performing the analogous computation in our idealized setting, we find this correlation generally to be negative. It turns positive, however, when shock persistence is very

high. This is illustrated in Table (2) below, again for a representative parameter set:

Table (2)

$$\text{corr}(\text{HP}_{t,n}^1 - r_{t,1}, r_{t,n} - r_{t,1})$$

$$\alpha = .25, \beta = .95, \gamma = -1$$

$$\pi = .33$$

$$\pi = .9$$

n = 2	-.806	.329
n = 3	-.949	-.0338
n = 4	-.968	.0242
n = 5	-.974	.0804
n = 10	-.983	.0668
n = 15	-.905	-.302

These results are consistent with the general expectations hypothesis that the interest rate is more likely to rise the steeper is the yield curve. For our economy, upward sloping and downward sloping yield curves, respectively, correspond to a historically low or high interest rate level. These correlations, therefore, are also consistent with the traditional Keynesian view. Campbell and Shiller (1984) had difficulties, however, in deriving such a negative correlation and suggested the possibility of a "perverse" relationship.

More traditional studies (c.f. Fama (1975)) have noted that while rate levels are highly correlated, interest rate changes are uncorrelated. In general this phenomenon is not confirmed in the model. The table below compares the correlations of successive and two period lagged rate changes for bonds on one and fifteen year maturities:

Table (3)

$$\text{corr}(r_{t+j,n} - r_{t+j-1,n}, r_{t,n} - r_{t-1,n})$$

$$\alpha = .25, \beta = .95, \pi = .7$$

$$\gamma = .5$$

$$\gamma = -1$$

	n = 1	15	1	15
j = 1	.187	.794	.213	.823
j = 2	.032	.494	.163	.560

What is especially striking is the high degree of persistence in rate changes on long bonds. Once again, this is a result of the ergodic property of the model: since so much of the yield-to-maturity on long bonds is due to events in the distant future -- the probability of which is unaffected by events today (ergodicity), then yields change little over the business cycle, thus giving rise to high correlations. That short interest rates are more closely related to the cycle than long rates -- or that long rates "underreact" to changes in the short rate -- is indicated by numerous studies; e.g., Campbell and Shiller (1984) and Mankiew and Summers (1984). Here -- unlike what seems to be the claim in some of these studies -- this property cannot be taken as an indication of a failure of a rational expectations theory of the interest rate (see also the comments by Weiss (1984)).

The power of the risk aversion parameter to smooth consumption is also evidenced by the fact that for the $\gamma = -1$ case, the two period lagged rate changes are still very high. We wish to point out, however, that these earlier studies which reported negligible correlation dealt with measured changes over intervals of time which are relatively very much shorter than those of this model (judging from the magnitude of the productivity shocks and the size of the discount factor β).

The high degree of persistence in the long rates should also produce persistence in their excess holding returns. This is confirmed by Table 4 where we have computed the serial correlation $\text{corr}(\text{HP}_{t+j,n}^1 - \text{HP}_{t+j,1}^1, \text{HP}_{t,n}^1 - \text{HP}_{t,1}^1)$ for different maturities and parameter values. We note the very high correlation when the shock persistence is high.

Table (4)

$$\text{corr}(\text{HP}_{t+j,n}^1 - r_{t+j,1}, \text{HP}_{t,n}^1 - r_{t,1})$$

$$\alpha = .25, \beta = .95, \gamma = 0$$

n:	2	3	4	5	10	15
$\pi = .5$	.472	.467	.463	.462	.461	.461
$\pi = .9$	.592	.753	.742	.750	.804	.844

$$\alpha = .25, \beta = .95, \gamma = -1$$

$\pi = .5$	.402	.511	.495	.490	.485	.485
$\pi = .9$	.472	.707	.727	.712	.779	.858

Thus, on average for this model, investors excess returns today on the ownership of long bonds will, to be a significant degree, be preserved in the future.

In particular, efficiency alone does not require that holding period returns be serially uncorrelated. To ensure such independence would apparently require a more specialized market structure than the one presented here. Indeed, the message of these results is that investors must use such information if they are to earn returns consistent with the level of risk they are bearing. This model suggests that the central characteristic of market efficiency is not that investors are unable to secure and use publicly available information but that in using such information they are unable to earn returns which are unjustified by the corresponding risk.

6. Issues of Information

A central issue regarding the term structure is the degree to which forward rates approximate future spot rates. Our results suggest that the term structure may not be a very useful tool for forecasting future rates. Indeed, for this model, the forward premium can be highly unstable. Once again, this is not a sign of market inefficiency but may rather simply reflect the absence of a sufficiently rich market structure in this simple model.

As the average yield curve is rising, forward rates, on average, must also provide biased forecasts of future spot rates (this is so as the expected spot rate is constant in a stationary economy) in this model. Equivalently, the average forward premium will be positive.

The general picture is as expected. (1) the longer is the forecast period (and the larger is the volatility of the forecasted interest rate), the larger is the forward premium, and (2) the longer is the maturity of the interest rate forecasted (and the smaller is its volatility), the smaller is the forward premium. It is also interesting to note the effect, presented in the lower half of Table (5), of a simultaneous increase in the economy's shock persistence and risk aversion. While the former change should lead to a decrease in the volatility of future interest rates, the latter change should increase the impact of a given volatility on the forward premium. We see that the former effect dominates for the shortest forecast span and maturity, while the latter dominates for the longest forecast spans. The net effect is thus to widen the spread of the forward premia across the maturity range.

Table 5

The Structure of the Forward Premium

$$(\beta = .95, \gamma = -1, \pi = .5)$$

n	n-period ahead forecast of the one period rate			1-period ahead forecast of the n-period rate		
	f_{t+n}^1	$E_t(r_{t+n,1})$	$FP_{t,n}^1$	f_{t+1}^n	$E_t(r_{t+1,n})$	$FP_{t,1}^n$
1	-.9%		13.0%	-.9%	-13.9%	13.0%
5	5.2%	-13.9%	19.1%	3.2%	-1.7%	4.9%
10	5.3%		19.2%	4.1%	1.3%	2.8%
15	5.3%		19.2%	4.4%	2.5%	1.9%

$$(\beta = .95, \gamma = -2, \pi = .9)$$

n	f_{t+n}^1	$E_t(r_{t+n,1})$	$FP_{t,n}^1$	f_{t+1}^n	$E_t(r_{t+1,n})$	$FP_{t,1}^n$
1	-17.66%		2.3%	-17.66%	-19.9%	2.3%
5	.5%	-19.9%	19.4%	-8.3	-12.4%	3.9%
10	4.2%		24.1%	-3.1%	-5.8%	2.7%
15	5.0%		25.0%	-.6%	-2.6%	2.0%

The usefulness of the yield-curve information regarding the future expected interest rates obviously depends upon the stability across time of the forward premia: if these premia are reasonably stable, then an adjustment of forward rates for their average values should yield useful forecasts of future interest rates. Table (6) provides us with an indication of this stability within our model economy. Here we provide (conditional) yield curve information at, respectively, the "top," "bottom," and "middle" of the cycle (the middle of the cycle represents the median capital stock-shock combination). This table gives forward and expected one period interest rates, and the forward premium $FP_{t,n}^1 = f_{t+n}^1 - E_t(r_{t+n,1})$ for various time periods ahead. We record the

following observations, some of which are immediate extensions of previous results:

Observation 6.1: In general, the forward premium is positive for all time periods and all points on the cycle.

Observation 6.2: With low shock persistence and low risk aversion, the forward premium for time periods distant in the future is stable over the cycle. For the immediate future, however, the premium varies considerably and positively with the level of interest rates. The premium is high when interest rates are high (at the "bottom of the cycle") and low when rates are low (at the "top of the cycle").

Observation 6.3: Higher shock correlation and greater risk aversion produce less stability of the forward premium at the short end of the term structure, and also reverse the relationship between the forward premium and the level of interest rates. Thus, the premium for all periods is high when rates are high and low when rates are low.

Thus, depending on the parameter choice, our model seems to accommodate both traditional, competing theories of the comovement of interest rates and forward premia. According to the Kessel (1965) liquidity interpretation of the forward premium, there should be a positive relationship between interest rates and forward premia as displayed by the upper part of Table (6). The Keynes-Hicks insurance view of the forward premium, on the other hand, predicts an inverse relationship with the interest rate level as displayed by the lower part of Table (6).

Table (6)

The Stability of the Forward Premium

$$(\alpha = .25, \beta = .95, \gamma = -1, \pi = .5)$$

	Bottom			Middle			Top (of Cycle)		
	f_{t+n}^1	$Er_{t+n,1}$	$FP_{t,n}^1$	f_{t+n}^1	$Er_{t+n,1}$	$FP_{t,n}^1$	f_{t+n}^1	$Er_{t+n,1}$	$FP_{t,n}^1$
n=1	52.6%	35.4%	17.2%	-13.9%	-24.5%	10.6%	-34.7%	-44.4%	9.7%
n=5	6.7%	-12.8%	19.5%	4.7%	-14.3%	19.0%	4.3%	-14.7%	19.0%
n=10	5.3%	-13.9%	19.2%	5.3%	-13.9%	19.2%	5.3%	-13.9%	19.2%
n=15	5.3%	-13.9%	19.2%	5.3%	-13.9%	19.2%	5.3%	-13.9%	19.2%

$$(\beta = .95, \gamma = -2, \pi = .9)$$

n=1	15.9%	15.1%	.8%	-22.7%	-21.4%	-1.2%	-44.5%	-51.4%	6.9%
n=5	13.6%	0.0%	13.6%	-5.2%	-21.1%	15.9%	-10.0%	-38.7%	28.7%
n=10	10.2%	-11.1%	21.3%	1.7%	-20.5%	22.2%	.6%	-28.3%	28.9%
n=15	7.8%	-16.0%	23.8%	3.8%	-20.2%	24.0%	3.4%	-23.6%	27.0%

We should also note that, while the upper part of the table contradicts the result of Startz (1982), this is not so for the lower part. Based on a simple version of a rational expectations structure, and using one-month t-bill data, he derives estimates of forward premium volatilities which increase monotonically with the length of the forecast span. The issue of forward premium obviously deserves further theoretical attention; in particular, the role played by risk aversion.

Fama (1984) suggests interest rate forecasts based upon forward rate differences as a possible way around the problem of highly volatile forward premia, at least for short term forecasting. He derives a surprisingly high correlation between the one-month forward-spot t-bill rate and the following month's change in the one-month rate. The analogous correlations $\text{corr}(r_{t+n+1,1} - r_{t+n,1}, f_{t+n+1}^1, -f_{t+n}^1)$ between later changes in this rate and the current spread between corresponding forward rates fall off dramatically, however, as the forecast horizon increases.

It is interesting to note that the identical phenomena is observed in our model. Table (7) below gives a representative description for the case $\alpha = .25$, $\beta = .95$, $\gamma = -1$, and $\pi = .5$

Table (7)

$\text{corr}(r_{t+n+1,1}^{-r_{t+n,1}}, f_{t+n+1}^1 - f_{t+n}^1)$	
$n = 1$	.763
$n = 2$	.129
$n = 3$	.089
$n = 4$	.030
$n = 5$	.013
$n = 10$	.000

Lastly we note that none of the qualitative (and, indeed, quantitative) results presented in Sections 5 and 6 are in any way substantially changed if the model's parameters are fixed in a manner similar to, e.g., Prescott (1986) or Hansen (1985).⁷

7. Comparative Dynamics

7.1 Changes in the Subjective Discount Factor

It is well known (see, e.g., Danthine and Donaldson (1981)) that an increase in the subjective discount factor β will cause the stationary distribution on capital stock not only to spread out, but also to shift to a range of higher capital values. Coincident with this is the following:

Observation 7.1: An increase in β will cause the (conditional) yield curve to decline in every state. As a consequence the average yield curve declines as well.

By causing agents to value the future more, an increase in β will engender an increase in demand for bonds (claims on future income) thereby increasing prices and reducing yields. This is reinforced by the effect of the above mentioned shift in the stationary capital stock distribution which reduces the return on the "market portfolio" -- $\int \int f'(k) \lambda H(dk, d\lambda)$ (see Donaldson and Mehra (1984) for an elaboration). By reducing returns on competing assets, bond demand is further increased.

It is difficult, however, to translate this intuition into mathematical formalism. This is due to the fact that, given an increase in β , we lack sufficient information regarding the transition path to the new steady state. Relying on known asymptotic properties, however, allows the following theorem:

Theorem 7.1: Suppose $\beta_1 < \beta_2$. For any $(k_i, \lambda_j) \in R^K(\beta_1) \times [\underline{\lambda}, \bar{\lambda}]$, and any $(k_s, \lambda_w) \in R^K(\beta_2) \times [\underline{\lambda}, \bar{\lambda}]$, there exists an N such that for $n \geq N$ $r_n(k_i, \lambda_j; \beta_1) > r_n(k_s, \lambda_w; \beta_2)$, where the conditioning on β_i has the obvious interpretation. Furthermore, $r_n(\beta_1) > r_n(\beta_2)$.

Proof: Since $r_n(k_i, \lambda_j; \beta_1) \rightarrow \frac{1}{\beta_1} - 1$, and $r_n(k_s, \lambda_w; \beta_2) \rightarrow \frac{1}{\beta_2} - 1$ where $\frac{1}{\beta_1} - 1 > \frac{1}{\beta_2} - 1$ as $n \rightarrow \infty$, such an N exists. An identical observation applies to $\{r_n(\beta_1)\}$ and $\{r_n(\beta_2)\}$. ■

7.2 Changes in Intertemporal Substitution

In our model context, a decrease in the value of the parameter γ suggests greater intertemporal substitution possibilities on the part of economic agents. As these agents are risk averse, this greater willingness to substitute consumption today for consumption tomorrow will give

rise to consumption smoothing. With a highly uncertain technology (and thus stock market), this desire for a less variable pattern of real consumption will lead to greater demand for risk free bonds, thereby bidding up their prices and lowering the term structure (conditional and average) -- except in the limit which is unrelated to γ . A simpler -- though more naive -- interpretation is simply to observe that γ is the measure of relative risk aversion (though strictly speaking only in a one-period setting), and more risk averse agents will increase their demand for risk free securities with the attendant consequences. This intuition is, in fact, confirmed by our numerical exercise.

Observation 7.2: A decrease in γ gives rise to an increase in the conditional and average yield curves.

It is also interesting to observe that the reduced variability of consumption is concomitant to an increase in the variability of interest rates. This is illustrated in Table (8) below.

This illustrates the notion that if "quantities" are constrained to be less variable, "prices" must vary more to achieve equilibrium.

Table (8)

Parameters $\alpha = .25$, $\beta = .95$, $\Pi = .5$, $\gamma \in \{.5, -1\}$

$\gamma = .5$

$\gamma = -1$

Standard Deviation of Stationary Consumption Distribution

.46678

.4224

Corr(c_{t+n} , c_t), Various n , Stationary Time Series

$n = 1$

.4223

.5449

$n = 2$

.1332

.2507

$n = 5$

.0333

.1102

Standard Deviation of Yield to Maturity, Maturity n

$n = 1$

.163

.400

$n = 2$

.123

.331

$n = 3$

.091

.266

$n = 4$

.070

.221

$n = 5$

.056

.184

$n = 10$

.028

.095

$n = 15$

.019

.064

7.3 Changes in Shock Correlation (π)

Just as for the measure of intertemporal substitution, the degree of shock persistence does not affect the yield curve asymptotically. As pointed out in Danthine et al. (1983), however, increasing persistence will increase the period-to-period autocorrelation in output, capital stock and consumption (see Danthine et al. (1983) for a discussion of the effect of such changes on optimal policies). From our perspective, the important fact is that the period to period variation in consumption is reduced; in essence, the economic environment becomes more certain.

Now the term structure under certainty (also under risk neutrality) is absolutely flat with uniform value $1/\beta - 1$. We would thus expect, on average, that an increase in persistence would tend to "flatten out" or reduce the slope of the term structure. This is exactly what is

observed. In Figure (1) below, we graph the average term structure for the case in which $\gamma = -1$, and $\beta = .95$. The qualitative properties thus illustrated are representative of what is observed for other cases.

The effect on the conditional term structure depends on the current state. If the current state is "good," i.e. a high (k, λ) pair, an increase in the persistence will cause the term structure to rise. The intuition behind this result is that with an increase in persistence agents expect the future to be prosperous and, *ceteris paribus*, agents will pay less for risk free pure discount bonds, thereby causing rates to rise. Consistent with our earlier remarks, it will also flatten out (see Figure (1) where the $\alpha = .25$, $\beta = .95$, $\gamma = -1$ case is presented).

At the other end of the spectrum, if the current state is "poor," an increase in persistence is likely to have the opposite effect. Expecting a bleak future, agents will bid up the price of bonds delivering certain consumption; hence, interest rates and consequently the yield curve will decline. This polar behavior is also confirmed by our simulation results. We summarize our results as follows:

Observation 7.3: An increase in persistence causes the average and conditional term structures to flatten out; that is, to have reduced slope.

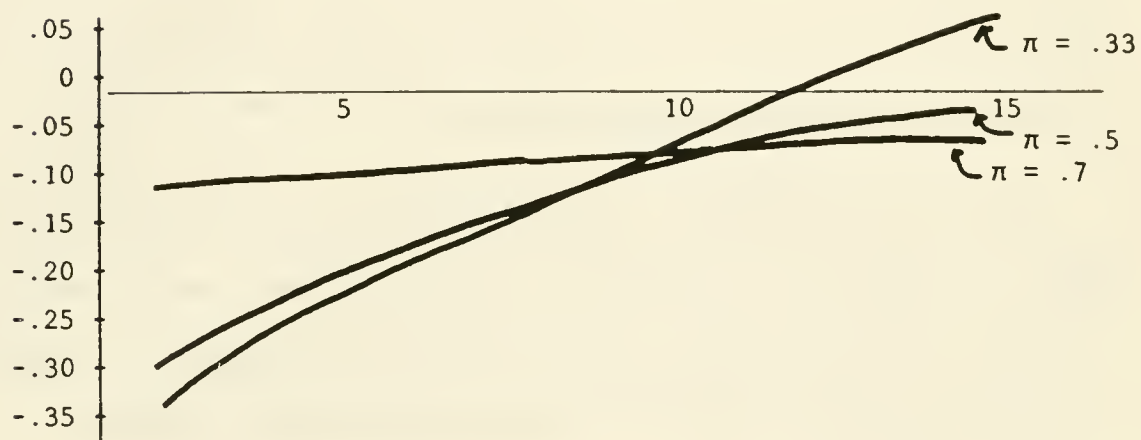
Although this increase in persistence is observed to increase the autocorrelation in consumption, it nevertheless increases the variance of the stationary consumption distribution as well. Campbell (1985), in a related model without production finds that real returns to real bonds fall as the variance on the consumption process increases. This, as well, is observed in our model -- though not asymptotically.

Figure 1

Changes in π

The Term Structure: Average of Conditional Yield Curves

$\delta = -1$, $\beta = .95$, various π



Clearly, government policy can affect the persistence of economic time series. Our results suggest that stabilization policies, if effective, should directly influence the real yield curve. In particular, on average, an increase in persistence would tend to reduce the slope (see figure (1)).

7.4 Changes in Capital Income Tax Rates

There is one additional issue that can easily be addressed in this framework and that is the influence of capital income taxation rates on the shape of the yield curve. Suppose that the representative consumer-investor undertakes consumption-savings decisions which satisfy the following problem:

$$\begin{aligned} \max E \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } c_0 + k_0 = a_0, \text{ given} \\ c_t + k_{t+1} = (1 - \theta)R_t k_t + w_t + g_t \end{aligned}$$

Here θ denotes the capital income tax rate, R_t the period t return to capital ($R_t = f'(k_t)\lambda_t$), w_t the wage rate ($w_t = f(k_t)\lambda_t - f'(k_t)\lambda_t k_t$), and g_t the lump sum subsidy equal in magnitude to the total tax revenue; i.e., $g_t = \theta R_t k_t$. It has been shown by Becker (1982) (in a world of certainty) and Danthine and Donaldson (1984) (for uncertainty) that the competitive allocations arising in this economy are identical to those that would arise in an economy where the competitive agents solve

$$\max E \left(\sum_{t=0}^{\infty} (\beta(1 - \theta))^t u(c_t) \right)$$

$$\text{s.t. } c_0 + k_0 = a_0, \text{ given}$$

$$c_t + k_{t+1} = f(k_t)\lambda_t$$

This equivalence clearly demonstrates that with respect to the time series of consumption, capital, and output in this type of model, an increase in the capital income tax rate is equivalent to a decrease in the discount factor of the representative agents. Thus an increase in the capital tax rates will affect the yield curve exactly as would a reduction in agents' discount factors in a world without taxation.

Applying the results of Section 7.1, we have the following:

Observation 7.4: An increase in the proportional tax rate will cause the conditional and average yield curves to rise.

This is expected. Although total income remains unchanged (due to the lump sum redistribution) the return to savings (whether as stocks or bonds) is reduced. Thus agents save less out of all output levels. Bond prices fall and the yield curve rises.

8. Concluding Comments

In this paper we have analyzed a simple RBC model with regard to its ability to replicate the qualitative properties of the term structure of interest rates. Our efforts have been hampered by the relative dearth of empirical studies of real interest rate behavior for bonds of long maturity (up to fifteen years) -- a fact that has forced us to undertake

our comparisons with studies of nominal rate behavior. Given this caveat, our model supports the empirical literature in a surprising number of ways. Least satisfying, perhaps, have been our results concerning the behavior of the yield curve over the cycle (and, indeed, even in this case our results are not categorically at variance with the literature). A worthwhile focus of further research would be to improve the model's performance in this regard.

We have not attempted to calibrate the model in the sense of seeking that set of parameters which gives interest rate behavior which best approximates reality quantitatively. Besides being at variance with the RBC tradition, such an attempt would be futile for reasonable parameter values: the economy is simply too productive. In this light we view our research as an appropriate springboard to future efforts by documenting modelling goals to be achieved.

Although we are unaware of any study which addresses the precise issues considered here, there are many papers which have adopted the general modelling perspective of infinitely lived representative agents. Brown and Gibbons (1985) study a closely related model but with the objective of estimating the coefficient of relative risk aversion from consumption/wealth data. Hansen and Singleton (1983) seek to characterize explicitly restrictions on the joint distribution of asset returns and consumption implied by the representative agent class of models and to obtain parameter estimates for preferences and the assumed stochastic consumption process. They deal with multiple goods but impose a joint distribution on consumption and asset returns, a relationship that is numerically calculated in our model. Dunn and Singleton (1986) investigate term structure relations implied by a model in which preferences are

non-separable functions of the service flow of two durable goods. Once again they focus on estimating preference parameters and parameters characterizing the assumed co-movements of consumption and Treasury bill returns. Lastly we mention Breeden (1986). Several of the results of Sections 4 and 7 are contained in his important piece. He, however, employs a continuous time methodology -- techniques which as yet seem inappropriate to business cycle studies.

Appendix 1

A.1 The production technology $f(\cdot)$ is increasing, bounded, strictly concave and three times continuously differentiable, with $f(0) = 0$, and $f'(0) = \infty$.

A.2 The period utility function $u(\cdot)$ is increasing, bounded, strictly concave, and three times continuously differentiable and displays constant or decreasing relative risk aversion. Furthermore, for any interval (ε, M) , where $\varepsilon > 0$, $M < \sup_k f(k)$, there is a W such that

$$\infty > W > \frac{u'''(\gamma_1)}{|u''(\gamma_2)|}, \quad \forall \gamma_1, \gamma_2 \text{ in } (\varepsilon, M).$$

A.3 The shock λ is subject to a stationary Markov process with strictly positive bounded support $[\underline{\lambda}, \bar{\lambda}]$, where $\underline{\lambda} > 0$. Furthermore, we require that $F(\cdot; \lambda')$ stochastically dominate $F(\cdot; \lambda)$ in the first degree (Rothschild and Stiglitz (1971)) whenever $\lambda' > \lambda$. (This requirement is intended to capture the notion that tomorrow is more likely to be similar to today than radically different.) Lastly, for all λ_1, λ_2 in the support of $F(\cdot; \cdot)$,

$$\int |F(d\lambda; \lambda_2) - F(d\lambda; \lambda_1)| < B|\lambda_2 - \lambda_1|,$$

$$\text{for } B = \frac{\lambda f(k_L)M}{2\beta M_1 M_2 \bar{\lambda}}.$$

The terms M, M_1, M_2 are defined, respectively, by

$$\sup_{c_L \leq c \leq c_U} (u''(c)), \quad \beta \max[f'(k_0), f'(k_L/2)] \text{ and } u'(c_L/2), \text{ where } [c_L, c_U]$$

and $[k_L, k_U]$ are intervals which bound the asymptotic consumption and capital paths. Since β , $\underline{\lambda}$, and $\bar{\lambda}$ alone determine these bounds, B is independent of $F(\cdot; \cdot)$. (This requires that if two production shocks today do not differ by very much, the conditional distributions of next period's shock must be closely similar.)

Theorem 2.1: Given A.1, A.2, and A.3 there exists a unique solution to (P); that is, a unique, bounded, increasing, concave value function $V(\cdot)$ such that

$$(1) \quad V(k, \lambda) = \max_{c \leq f(k)\lambda} \{u(c) + \beta \int V(f(k)\lambda - c)\lambda', \lambda') F(d\lambda'; \lambda)\}$$

where $V(k, \lambda)$ is the maximum attainable expected utility for all feasible consumption paths, and λ' represents next period's shock. Furthermore, there exists a unique, bounded, continuously differentiable, increasing function $c(k, \lambda)$ such that

$$(2) \quad V(k, \lambda) = u(c(k, \lambda)) + \beta \int V(f(k)\lambda - c(k, \lambda), \lambda') F(d\lambda'; \lambda).$$

Here $c(k, \lambda)$ is the optimal consumption policy. The optimal savings policy $s(k, \lambda)$ is thus $f(k)\lambda - c(k, \lambda) = s(k, \lambda)$. Lastly, the concavity of $V(\cdot, \cdot)$ ensures that a necessary and sufficient characterization of the optimal consumption policy is given by:

$$(3) \quad u'(c(k, \lambda)) = \beta f'(s(k, \lambda)) \int u'(c(s(k, \lambda), \lambda')) F(d\lambda'; \lambda)$$

Proof: Donaldson and Mehra (1983, Theorem 2.1).

Theorem 2.2: Given assumptions A.1, A.2, and A.3, the policy functions $c(\cdot, \cdot)$ and $s(\cdot, \cdot)$ define Markov processes on capital stock and

consumption. Furthermore, these processes possess unique invariant measures, which describe their long run average behavior.

Proof: Donaldson and Mehra (1983) or Futia (1982).

Appendix 2

To illustrate this procedure, define the set T_{ij} by

$$(A.2.1) \quad T_{ij} = \{(k_i, \lambda_j) : k_i \in \Gamma, \lambda_j \in \{\frac{1}{2}, 1, 3/2\}\}$$

For each $(k_i, \lambda_j) \in T_{ij}$ first compute the table T_0 of values where

$$(A.2.2) \quad T_0 = \{V_0(k_i, \lambda_j) : V_0(k_i, \lambda_j) = u(f(k_i)\lambda_j); (k_i, \lambda_j) \in T_{ij}\}$$

In the next stage of this process, construct two tables, the first of which, T_1 , being defined by:

$$(A.2.3) \quad T_1 = \{V_1(k_i, \lambda_j) : V_1(k_i, \lambda_j) = \max_{\substack{s_1(k_i, \lambda_j) \in \Gamma \\ s_1(k_i, \lambda_j) \leq f(k_i)\lambda_j}} \{u(f(k_i)\lambda_j - s(k_i, \lambda_j)) \\ + \beta \sum_{\ell=1}^3 V_0(s(k_i, \lambda_j), \lambda_\ell) \Pi_{j\ell}\} : (k_i, \lambda_j) \in T_{ij}\}$$

The second table S_1 , records the optimal savings level which solves $V_1(\cdot, \cdot)$; i.e.

$$(A.2.4) \quad S_1 = \{s_1(k_i, \lambda_j) : \forall (k_i, \lambda_j) \in T_{ij} \quad s_1(k_i, \lambda_j) \text{ is the solution to } V_1(k_i, \lambda_j)\}.$$

By repeating this process over and over again we were able to define the optimal savings policy $s^*(k_i, \lambda_j)$ for all $(k_i, \lambda_j) \in T_{ij}$ as

$$(A.2.5) \quad s^*(k_i, \lambda_j) = \lim_{n \rightarrow \infty} s_n(k_i, \lambda_j);$$

analogously, the optimal consumption function is given by $c^*(k_i, \lambda_j) = f(k_i)\lambda_j - s^*(k_i, \lambda_j)$.

The entries $\tilde{\pi}_{(i,j),(s,w)}$ which give $\text{Prob}((k_{t+1}, \lambda_{t+1}) = (\hat{k}_s, \lambda_w) \mid (k_t, \lambda_t) = (\hat{k}_i, \lambda_j))$ are governed by the optimal savings policy in the following way:

$$(A.2.7) \quad \tilde{\pi}_{(i,j),(s,w)} = \begin{cases} 0 & \text{if } s^*(\hat{k}_i, \lambda_j) \neq \hat{k}_s \\ \pi_{jw} & \text{if } s^*(\hat{k}_i, \lambda_j) = \hat{k}_s. \end{cases}$$

Powers of the matrix $\hat{\pi}$ give the n -step transition probabilities; i.e., entry $\tilde{\pi}_{(i,j),(s,w)}^n$ of $(\hat{\pi})^n$ describes $\text{Prob}((k_{t+n}, \lambda_{t+n}) = (\hat{k}_s, \lambda_w) \mid (k_t, \lambda_t) = (\hat{k}_i, \lambda_j))$. At this point all the information is available for calculating the (approximate) term structure.

The general formula for pricing a pure discount bond must first be obtained. In this setting a pure discount bond issued in state $(\hat{k}_i, \lambda_j)$ and paying one unit of consumption in period n irrespective of the state must, in equilibrium, be priced according to

$$(A.2.8) \quad P_n(\hat{k}_i, \lambda_j) = \beta^n \sum_{(k_s, \lambda_w) = (\hat{k}_1, \lambda_1)}^{(k_s, \lambda_w) = (\hat{k}_n, \lambda_n)} \left[\frac{u'(c^*(\hat{k}_s, \lambda_w))}{u'(c^*(\hat{k}_i, \lambda_j))} \right] \tilde{\pi}_{(i,j),(s,w)}^n.$$

These are the "conditional" bond prices. For every state $(\hat{k}_i, \lambda_j) \in \hat{K} \times \{\frac{1}{2}, 1, 3/2\}$, the analogous state dependent term structure (we computed it for 15 time periods) is thus defined by the sequence

$$\left\{ \frac{1}{P_1(\hat{k}_i, \lambda_j)} - 1, \left(\frac{1}{P_2(\hat{k}_i, \lambda_j)} \right)^{\frac{1}{2}} - 1, \left(\frac{1}{P_3(\hat{k}_i, \lambda_j)} \right)^{\frac{1}{3}} - 1, \dots, \left(\frac{1}{P_{15}(\hat{k}_i, \lambda_j)} \right)^{\frac{1}{15}} - 1 \right\}.$$

To compute average (across all states) bond prices, it is first necessary to determine the limiting relative frequency of the various $(\hat{k}_i, \lambda_j)$ pairs. The relative frequency with which the individual $(\hat{k}_i, \lambda_j)$

pairs appears in the sequence $S(K, \lambda)$ reasonably approximates this true distribution. Thus the average price of an n -period pure discount bond was approximated according to

$$(A.2.9) \quad \hat{P}_n = \frac{1}{40,000} \sum_{(k_t, \lambda_t) \in S(K, \lambda)} P_n(\hat{k}_t, \lambda_t).$$

The term structure as computed from average bond prices $\{\hat{r}_n\}$ was then determined according to $\{\frac{1}{P_1} - 1, (\frac{1}{P_2})^{\frac{1}{2}} - 1, (\frac{1}{P_3})^{\frac{1}{3}} - 1, \dots, (\frac{1}{P_{15}})^{\frac{1}{15}} - 1\}$. Alternatively, the term structure as the average of conditional interest rates, $\{r_n\}$ was derived as

$$r_n = \frac{1}{40,000} \sum_{(k_t, \lambda_t) \in S(K, \lambda)} \left[\frac{1}{P_{R_F}^n(k_t, \lambda_t)} \right]^{\frac{1}{n}} - 1.$$

Footnotes

¹This model has been studied in Brock and Mirman (1972, 1973), Levhari and Srinivasan (1969), Mirman and Zilcha (1975), Mirrlees (1974) and others when the shocks to technology are i.i.d. Donaldson and Mehra (1983) consider the case when the shocks are correlated.

²For a full equivalence, even in the case of non-stationary equilibria, a certain transversality condition at infinity must also be satisfied. This issue is fully detailed in works by Araujo and Scheinkman (1983), Benveniste and Scheinkman (1982) and Weitzman (1973).

In addition to Prescott and Mehra (1980), Brock (1982) has developed a general equilibrium model with production that characterizes equilibrium in financial markets. Other related work includes the continuous time model of Cox, Ingersoll, and Ross (1980).

³Few topics in financial economics have as long been the subject of continuous research interest as the term structure of interest rates. Indeed, this literature goes back nearly a century and is too extensive to be fully detailed here. Fisher (1896), Hicks (1946) and Meiselman (1962) are some of the more prominent earlier researchers. More recent theoretical studies include the works of Cox, Ingersoll and Ross (1980, 1981), Le Roy (1982, 1984(i,ii)), Richard (1978), Singleton (1979), and Shiller (1979). Le Roy (1982, 1984(i,ii)), Singleton (1979), and Shiller (1979) utilize discrete time methodology while Richard (1978) and Cox et al. (1980, 1981) conduct their analysis in continuous time. Much of this work evaluates various forms of the unbiased expectations hypothesis.

⁴There is a third notion of The Term Structure as Computed from Average Bond Prices: Let $H(dk, d\lambda)$ denote the joint stationary distribu-

tion on capital stock and the productivity shock. Averaged across all states, n period pure discount bonds would have period t price

$$(7) \quad \hat{P}_n = \int P_n(k_t, \lambda_t) H(dk_t, d\lambda_t)$$

The analogous term structure $\{\hat{r}_n\}$ would thus be defined according to

$$(1 + \hat{r}_n)^n = \frac{1}{\hat{P}_n}.$$

Generally speaking, $\{r_n\}$ and $\{\hat{r}_n\}$ display the same qualitative properties.

⁵We do not claim that this is a realistic model of the business cycle, however.

⁶More precisely, these authors perform the equivalent correlations between the forward-expected spot rate differential and the long-short spread.

⁷In particular, following Prescott (1986) we fixed $\beta = .96$, $\alpha = .36$, $\gamma = .33$, $\Pi = .9667$ and $\lambda_t \in \{1.028, 1.0, .972\}$. These latter parameters induced a shock process which matches the one described in Prescott (1986) with regard to mean, standard deviation, and first order serial correlation. There remains the fact that our model allows complete depreciation while Prescott's (1986) does not. The effect of introducing of partial depreciation is further to smooth the series but not to alter the results of sections 5 and 6 (and 7, to follow).

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